International Journal of Theoretical Physics, Vol. 28, No. 4, 1989

Accelerating Kortweg–de Vries Solitons

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Received December, 9, 1988; revised January 5, 1989

A one-parameter transformation group which leaves a form of nonlinear Kortweg–de Vries equation invariant has been determined through a classical approach. Solutions of this differential equation have been obtained. They show the existence of accelerating solitons.

1. INTRODUCTION

The study of the solutions of differential equations by means of a one-parameter transformation group which retains the invariance of the differential equation was initiated by Lie (1874). The main idea behind this group approach is that if a one-parameter group of transformations keeps a differential equation unaltered, then the use of an invariant of the group results in reducing the original differential equation to a simplified form so that a solution may be determined.

A comprehensive account of work in this direction is found in Cohen (1911), Dickson (1924), Bluman and Cole (1974), Chester (1977), and Hill (1982).

The Korteweg-de Vries (KdV) equation is a well-known nonlinear partial differential equation occurring in many branches of physics, such as shallow water waves, lattice recurrences, plasma ion acoustic waves, and magnetohydrodynamics. Extensive information on KdV equation can be obtained in the work of Abolowitz and Segur (1979), Miles (1981), Knickerbocker and Newell (1980), and Bulloughs and Caudrey (1980).

An equation of the type

$$\frac{\partial c}{\partial t} - D_0 c\frac{\partial c}{\partial x} + \frac{\partial^3 c}{\partial x^3} = Q(t) \tag{1}$$

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0020-7748/89/0400-0481$06.00/0

considered in this problem occurs in the action of external periodic forces on a KdV soliton in the surface wave of the oceanic problem (Kundu, 1986). In this investigation we show that the solution obtained after interaction comes out as a periodic wave together with a soliton in an accelerating frame. Unlike the solitons obtained previously, this one moves with an acceleration. To present a solution in more detail, we have adopted the procedure of constructing a one-parameter transformation group which keeps the equation invariant.

2. BASIC METHOD OF CONSTRUCTING THE INFINITESIMAL FORM OF THE GROUP

The basic differential equation (1) involving one dependent variable c and two independent variables x and t can be kept invariant under the one-parameter transformation group

$$\begin{aligned} x_1 &= x + \varepsilon\xi(x, t, c) + O(\varepsilon^2) \\ t_1 &= t + \varepsilon\eta(x, t, c) + O(\varepsilon^2) \\ c_1 &= c + \varepsilon\zeta(x, t, c) + O(\varepsilon^2) \end{aligned} \tag{2}$$

For known functions ξ, η, and ζ the similarity variable and functional form of the solution are obtained by solving the first-order partial differential equation

$$\xi(x, t, c)\frac{\partial c}{\partial x} + \eta(x, t, c)\frac{\partial c}{\partial t} = \zeta(x, t, c) \tag{3}$$

The infinitesimal versions of the partial derivatives $\partial c/\partial x$, $\partial c/\partial t$, $\partial^2 c/\partial x^2$, and $\partial^3 c/\partial x^3$ are

$$\begin{aligned} \frac{\partial c_1}{\partial x_1} &= \frac{\partial c}{\partial x} + \varepsilon\pi_1 + O(\varepsilon^2) \\ \frac{\partial c_1}{\partial t_1} &= \frac{\partial c}{\partial t} + \varepsilon\pi_2 + O(\varepsilon^2) \end{aligned} \tag{4}$$

$$\frac{\partial^2 c_1}{\partial x_1^2} = \frac{\partial^2 c}{\partial x^2} + \varepsilon\phi_1 + O(\varepsilon^2) \tag{5}$$

$$\frac{\partial^3 c_1}{\partial x_1^3} = \frac{\partial^3 c}{\partial x^3} + \varepsilon\psi_1 + O(\varepsilon^2) \tag{6}$$

where

$$\begin{aligned} \pi_1 &= \zeta_x + (\zeta_c - \xi_x)\frac{\partial c}{\partial x} - \eta_x\frac{\partial c}{\partial t} - \xi_c\left(\frac{\partial c}{\partial x}\right)^2 - \eta_c\frac{\partial c}{\partial t}\frac{\partial c}{\partial x} \\ \pi_2 &= \zeta_t + (\zeta_c - \eta_t)\frac{\partial c}{\partial t} - \xi_t\frac{\partial c}{\partial x} - \eta_c\left(\frac{\partial c}{\partial t}\right)^2 - \xi_c\frac{\partial c}{\partial x}\frac{\partial c}{\partial t} \end{aligned} \tag{7}$$

$$\phi_1 = \pi_{1x} + \pi_{1c}\frac{\partial c}{\partial x} - \left(\xi_x + \xi_c \frac{\partial c}{\partial x}\right)\frac{\partial^2 c}{\partial x^2} - \left(\eta_x + \eta_c \frac{\partial c}{\partial x}\right)\frac{\partial^2 c}{\partial x\,\partial t} \tag{8}$$

$$\psi_1 = \phi_{1x}x + \phi_{1c}\frac{\partial c}{\partial x} - \left(\xi_x + \xi_c \frac{\partial c}{\partial x}\right)\frac{\partial^3 c}{\partial x^3} - \eta_x \frac{\partial^3 c}{\partial x^2\,\partial t} - \eta_c \frac{\partial c}{\partial x}\frac{\partial^3 c}{\partial x^2\,\partial t} \tag{9}$$

Since (2) leaves (1) invariant, i.e., we have

$$\frac{\partial c_1}{\partial t_1} - D_0 C_1 \frac{\partial c_1}{\partial x_1} + \frac{\partial^3 c_1}{\partial x_1^3} = Q(t_1) \tag{10}$$

Using (2), (4), and (6) in (10) and eliminating $\partial^3 c/\partial x^3$ by means of equation (1), we equate the coefficients of c and its partial derivatives of different order with respect to x and t to zero. These lead to

$$\begin{gathered}\eta_c = 0, \qquad \eta_x = 0, \qquad \xi_c = 0, \qquad \zeta_{cc} = 0 \\ \zeta_{xc} = \xi_{xx}, \qquad \eta_t = 3\xi_x \\ -\xi_t + D_0 c(2\zeta_c - 4\xi_x) - D_0\zeta + 3\zeta_{xxc} - \xi_{xxx} = 0 \\ \zeta_t - D_0\zeta_x c + \zeta_{xxx} + (\zeta_c - 3\xi_x)Q = \eta\frac{\partial Q}{\partial t}\end{gathered} \tag{11}$$

From (11) we finally deduce the infinitesimal form of the group

$$\begin{aligned}\xi &= \frac{E_0}{3}x - D_0E_0 t V(t) + \tfrac{7}{3}D_0E_0\int_{t_0}^{t} V(\lambda)\,d\lambda \\ &\quad - D_0E_1V(t) - D_0E_3 t + E_4 \\ \eta &= E_0 t + E_1 \\ \zeta &= \tfrac{4}{3}E_0 c - \tfrac{4}{3}E_0V(t) + E_0 t\frac{dV}{dt} + E_1\frac{dV}{dt} + E_3\end{aligned} \tag{12}$$

where we have set $Q(t) = dV/dt$.

3. GLOBAL FORM OF GROUPS AND SIMILARITY SOLUTION OF THE EQUATION

Using (12), the one-parameter group is obtained by solving

$$\begin{aligned}\frac{dx_1}{d\varepsilon} &= \frac{E_0}{3}x_1 - D_0E_0t_1V(t_1) + \tfrac{7}{3}D_0E_0\int_{t_0}^{t_1} V(\lambda)\,d\lambda \\ &\quad - D_0E_1V(t_1) - D_0E_3t_1 + E_4\end{aligned} \tag{13a}$$

$$\frac{dt_1}{d\varepsilon} = E_0 t_1 + E_1 \tag{13b}$$

$$\frac{dc_1}{d\varepsilon} = \tfrac{4}{3}E_0 c_1 - \tfrac{4}{3}E_0 V(t_1) + E_0 t_1\frac{dV}{dt_1} + E_1\frac{dV}{dt_1} + E_3 \tag{13c}$$

subject to initial conditions

$$x_1 = x, \qquad t_1 = t, \qquad c_1 = c \qquad \text{when} \quad \varepsilon = 0$$

Now we choose $E_0 = 0$. From (13b) we find

$$t_1 = t + \varepsilon E_1 \tag{14a}$$

From (13a) and (14a) we find

$$x_1 = x - D_0 E_1 \int_0^\varepsilon V(t + \varepsilon E_1)\, d\varepsilon - D_0 E_3 t\varepsilon - \frac{D_0 E_3 E_1}{2}\varepsilon^2 + E_4 \varepsilon \tag{14b}$$

Finally, using (14a) and (14b) in (13c), we find

$$c_1 = c + E_3 \varepsilon + E_1 \int_0^\varepsilon \frac{dV(t + \varepsilon E_1)}{dt}\, d\varepsilon \tag{14c}$$

In order to determine the functional form of the corresponding similarity solution from (3), we find

$$[-D_0 E_1 V(t) - D_0 E_3 t + E_4]\frac{\partial c}{\partial x} + E_1 \frac{\partial c}{\partial t} = E_1 \frac{\partial V}{\partial t} + E_3 \tag{15}$$

On solving this equation, we find that

$$Z = x + D_0 \int_{t_0}^{t} V(\lambda)\, d\lambda + \frac{D_0 E_3}{2E_1} t^2 - \frac{E_4}{E_1} t \tag{16a}$$

$$C = \chi(Z) + V(t) + aZ + \frac{E_3}{E_1} t \tag{16b}$$

Equations (16a) and (16b) are known as global invariants of equation (1).

When $E_3 = 0$, on substituting (16a) and (16b) in (1), we find

$$\frac{d^3\chi}{dZ^3} - \left(D_0 aZ + \frac{E_4}{E_1}\right)\left(a + \frac{d\chi}{dZ}\right) - D_0 \chi\left(\frac{d\chi}{dZ} + a\right) = 0 \tag{17}$$

Let us distinguish two cases.

Case I. Setting

$$\beta = D_0 aZ + D_0 \chi + E_4 / E_1$$

we get

$$d^3\beta / dZ^3 = \beta\, d\beta / dZ \tag{18}$$

Integrating, we get

$$\frac{d^2\beta}{dZ^2} = \frac{\beta^2}{2} + A_1 \tag{19}$$

Further integration yields

$$\left(\frac{d\beta^2}{dZ}\right)=\frac{\beta^3}{3}+2A_1\beta+A_2$$
$$\int\frac{d\beta}{(\beta^2+6A_1\beta+3A_2)^{1/2}}=\frac{1}{\sqrt{3}}\int dZ \tag{20}$$

In the special case when β and its derivatives tend to zero at ∞, $A_1=0=A_2$.
Integrating, we get

$$\beta=12/Z^2$$

Hence we find

$$\chi(Z)=\frac{12}{D_0Z^2}-aZ-\frac{E_4}{E_1D_0}$$

From (16b) we can write

$$c=V(t)+\frac{12}{D_0Z^2}-\frac{E_4}{E_1D_0} \tag{21}$$

Case II. When $a=0$, equation (17) reduces to the form

$$\frac{d^3\chi}{dZ^3}-D_0\chi\frac{d\chi}{dZ}-\frac{E_4}{E_1}\frac{d\chi}{dZ}=0 \tag{22}$$

Integrating, we get

$$\left(\frac{d\chi}{dZ}\right)^2=\frac{D_0}{3}\chi^3+\frac{E_4}{E_1}\chi^2+2B_1\chi+B_2 \tag{23}$$

Again in the special case when χ and its derivatives tend to zero at ∞, i.e., $B_1=B_2=0$, equation (23) has a cnoidal wave solution of the form

$$\chi=Cn^2Z \tag{24}$$

subject to the condition that

$$D_0=-12k^4, \qquad E_4=4E_1k^4$$

where k is called the modulus of the C_n function. (Bowman, 1961). In the degenerate case when $k\to 1$, solution (24) leads to the Soliton solution

$$\chi=\text{Sech}^2 Z \tag{25}$$

From (16b), combining with (24) and (25), respectively, we can write

$$c=V(t)+Cn^2Z \tag{26a}$$

$$c=V(t)+\text{Sech}^2Z \tag{26b}$$

4. DISCUSSION

The solutions obtained in (21), (26a) and (26b) apply to some special cases. In all these cases we have set $E_3 = 0$, which implies a space coordinate

$$Z = x + D_0 \int_{t_0}^{t} V(\lambda)\, d\lambda - \frac{E_4}{E_1} t$$

In this frame, the solutions (26a) and (26b) represent two traveling waves, one a cnoidal wave and the other a degenerate-state soliton. When $Q(t) = 0$, we find a traveling wave solution in a frame moving with constant velocity. When $Q(t) = dv/dt \neq 0$, we find that this frame at time t has velocity $D_0 V(t) - E_4/E_1$ and acceleration $D_0 V(t)$, which clearly represent a moving frame with nonuniform acceleration. The interesting features of the solutions (26a), (26b) are that, due to the existence of an external periodic or nonperiodic time-varying forcing term, the general solution consists of two parts. One is a term proportional to the forcing term and the other is a cnoidal or soliton solution in a moving frame whose acceleration is proportional to the time integral of the forcing term.

In oceanic studies such a KdV soliton may retain its identity, but begins to accelerate due to the external time-varying forcing term.

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